ESS 265: Instrumentation, Data Processing and Data Analysis in Space Physics

Lecture 15: Inverting Magnetic Field Data

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Model Inversions

- Often desirable to obtain a mathematical model of a set of observations.
- This model is usually an equation with a set of coefficients. If there are sufficient observations obtained at enough times or locations, then we can solve for the coefficients based on the observations. This process is called inversion.
- We will examine two quite different techniques.
  - Singular Value Decomposition
    - Used for geomagnetic field studies
  - Downhill Simplex
    - Used for complex models such as flux ropes

The Geomagnetic Field

- The geomagnetic field is usually represented in spherical polar coordinates (*r, θ, Ø*), with *θ* the angle from the polar axis (rotational) and Ø the clock angle (longitude) around the polar axis. The three components of the field can then be represented by a series of associated Legendre polynomials of degree *n* and order *m*. Here we show them in their usual Schmidt quasi-normalized form.

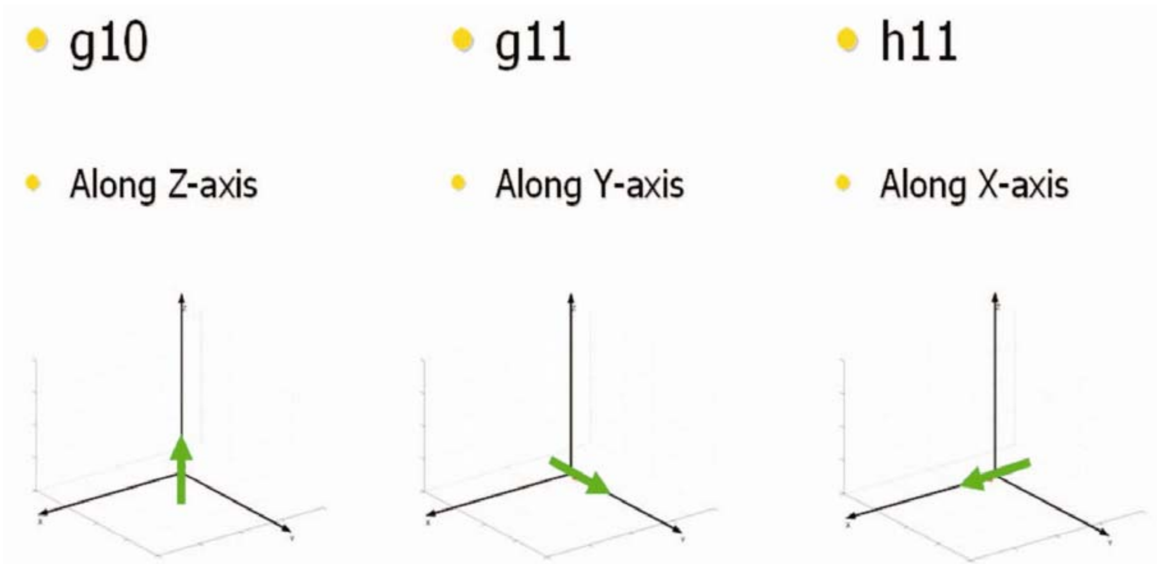
$$B_r = -\frac{\partial V}{\partial r} = \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[(n+1)\left(\frac{a}{r}\right)^{(n+2)}\left[g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right]\right] P_n^m(\cos\theta)$$

$$B_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[\left(\frac{a}{r}\right)^{(n+2)}\left[g_n^m \cos(m\phi) + h_n^m \sin(m\phi)\right]\right]\frac{dP_n^m(\cos\theta)}{d\theta}$$

$$B_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = \frac{1}{\sin\theta}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[m\left(\frac{a}{r}\right)^{(n+2)}\left[g_n^m \sin(m\phi) - h_n^m \cos(m\phi)\right]\right] P_n^m(\cos\theta)$$

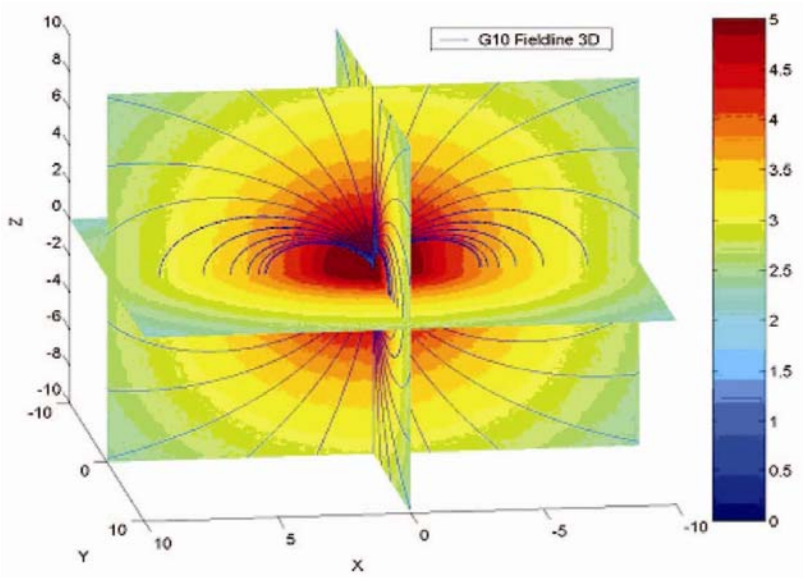
Dipole Moment

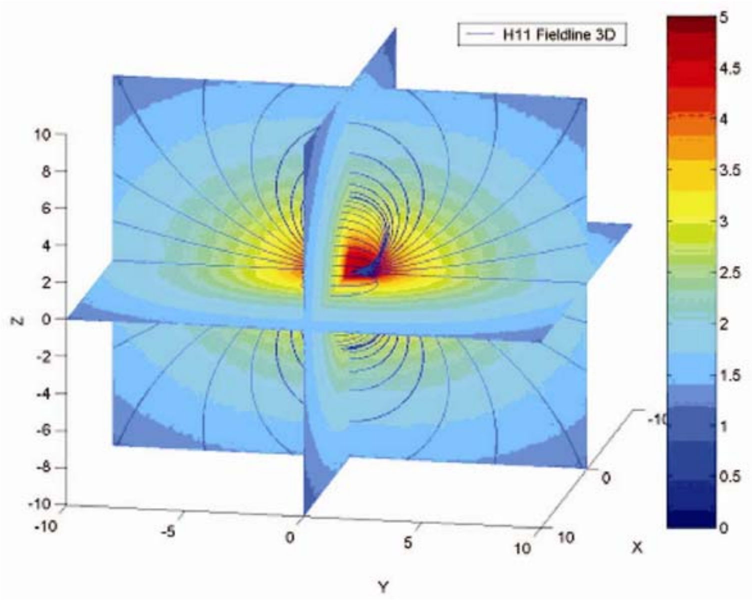
- A dipole is made from two opposite charges, *q* and –*q*, separated by a distance, *d*, apart. While electric charges can exist separately, magnetic charges cannot be separated or *div B=0* will be violated.
- A dipole can point in any one of three orthogonal directions or equivalently you can decompose any arbitrary dipole moment into three orthogonal components.
- It is traditional to designate the strengths of these three components by g_{10}, g_{11}, and h_{11}, where the leading number is the degree, *n*, and the trailing number is the order, *m*. The order, *m*, is always less than or equal *n*.



Dipole Moment (Continued)

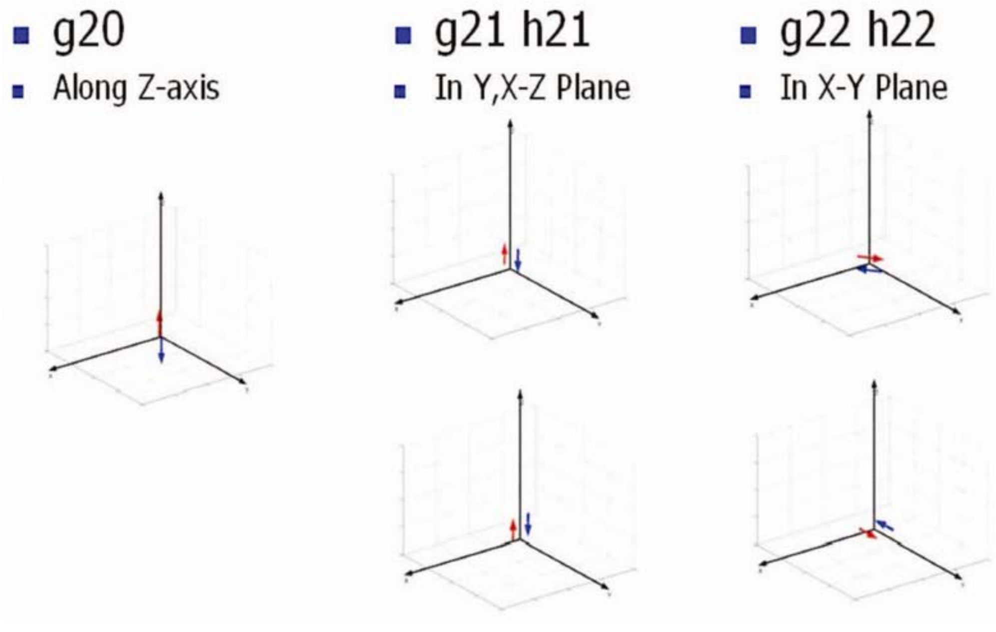
- The g_{10} term is rotationally symmetric about the polar axis.
- The strength of the field over the pole is twice that at the equator at the same distance.
- The field falls of as r^{-3}.
- The h_{11} (shown here) and the g_{11} terms are just rotated versions of the g_{10} term.
- The pole of g_{11} is along y and of h_{11} is along x.





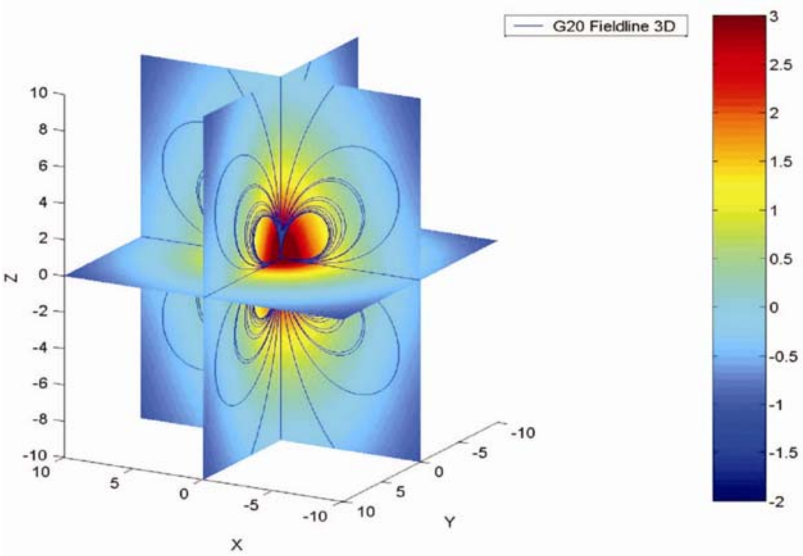
Quadrupole Moment 1

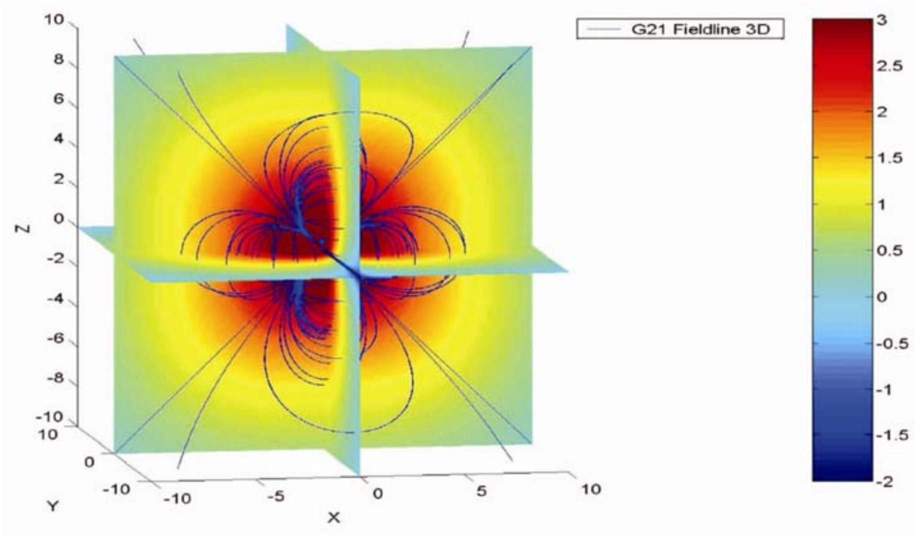
- A quadrupole moment can be constructed from two separated dipole moments a distance *d* apart.
- In three dimensions, there are three ways to separate charges to make a dipole moment, but there are five ways to separate dipole moments to make different quadrupoles.
- Three of the ways are separations of oppositely z-directed dipoles with separations in each of the three different coordinate axes.
- Two of the ways are moments in the x-y plane with separations in the x-y plane. Note that these two separation directions are not orthogonal. One is along *x* and one is at 45° to *x* and *y*.

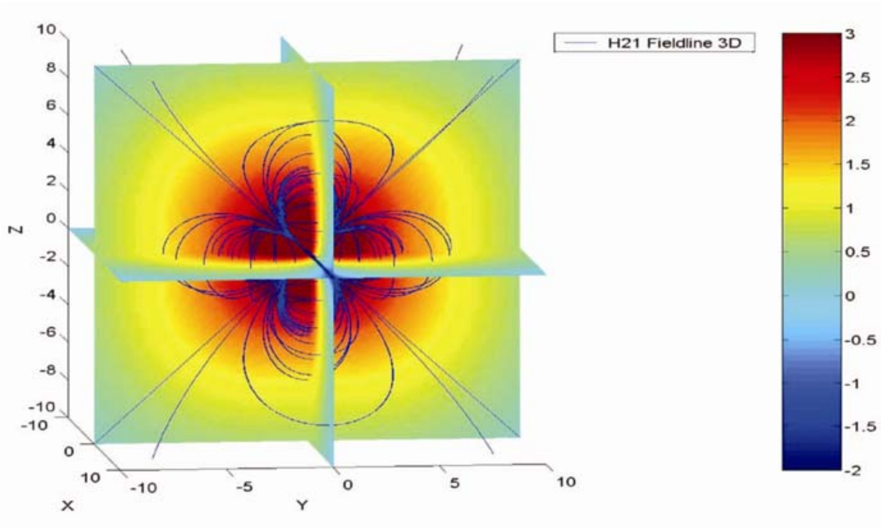


Quadrupole Moments 2

- The g_{20} term (like the g_{10} term) is rotationally symmetric about the polar axis, *z*.
- The g_{21} term and the h_{21} term are not rotationally symmetric, but are simply 90° rotations of each other about *z*. Thus, these are the two quadrupoles formed by dipole separations along *x* and along *y*.
- Note that unlike dipole fields, quadrupole fields do not necessarily remain in a plane.

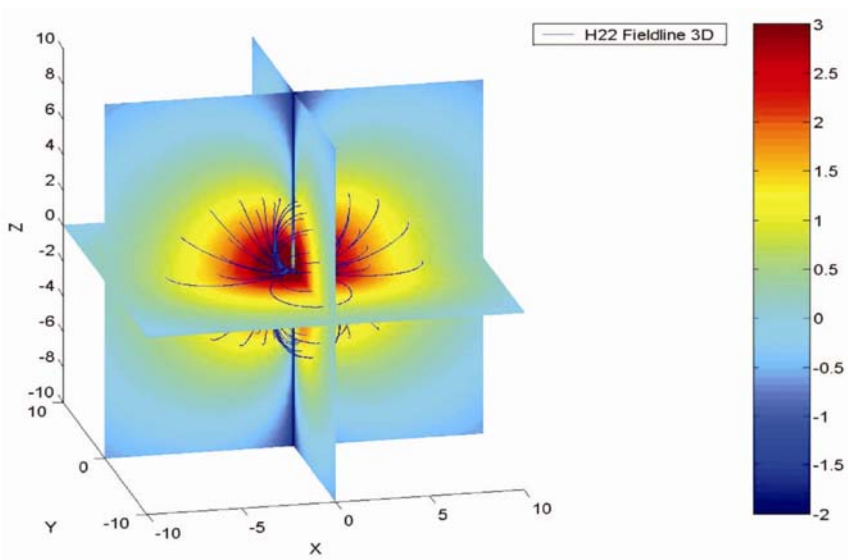


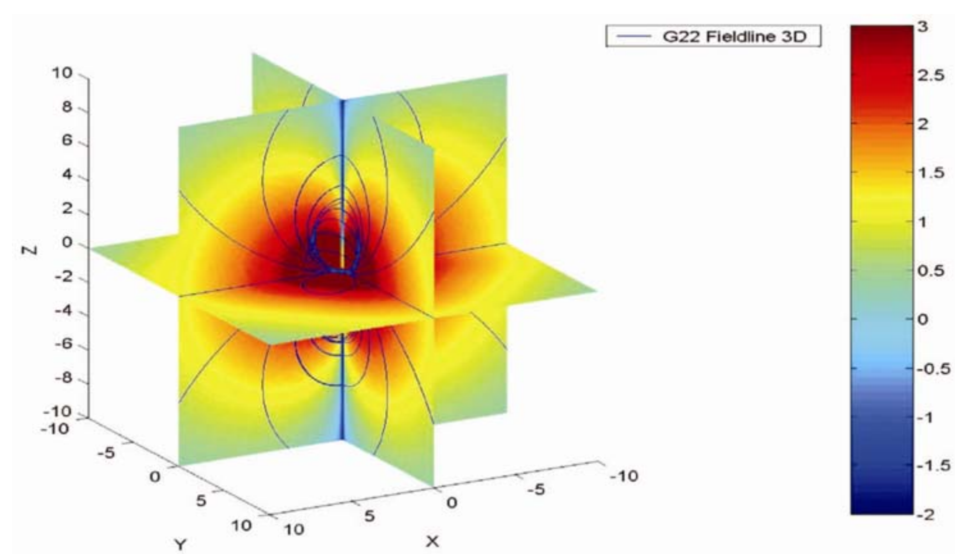




Quadrupole Moments 3

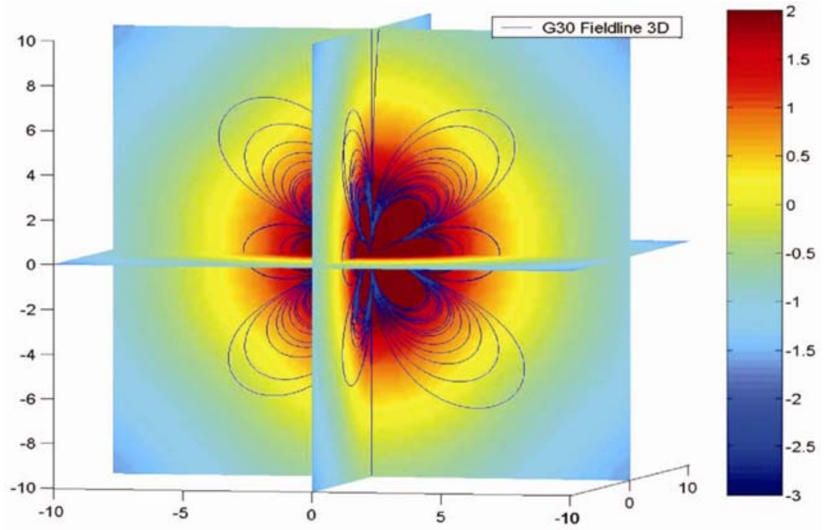
- The h_{22} moment as expected can be obtained from the h_{21} moment by a 90° rotation about x or from the g_{21} moment by two rotations: one about y and then one about z.
- The g_{22} moment can be obtained from h_{22} by a 45° rotation about z.
- Note that both h_{22} and g_{22} have a zero strength axis along z.

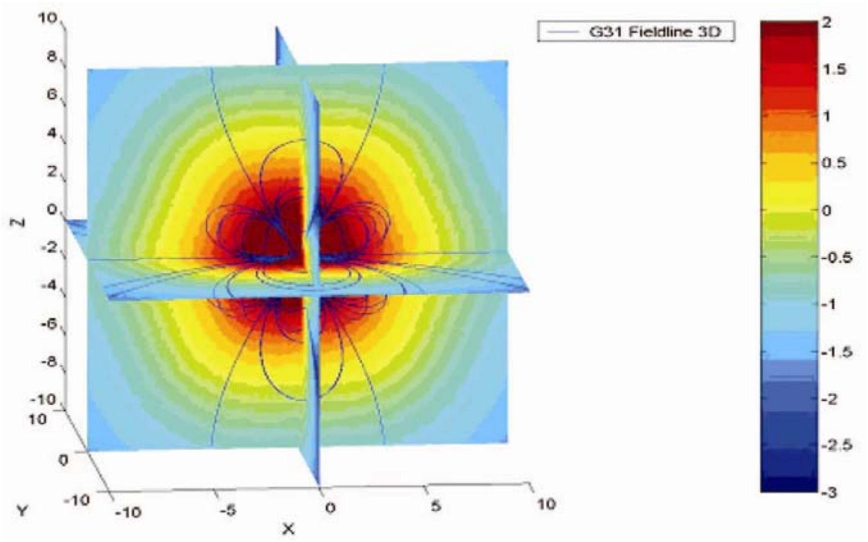


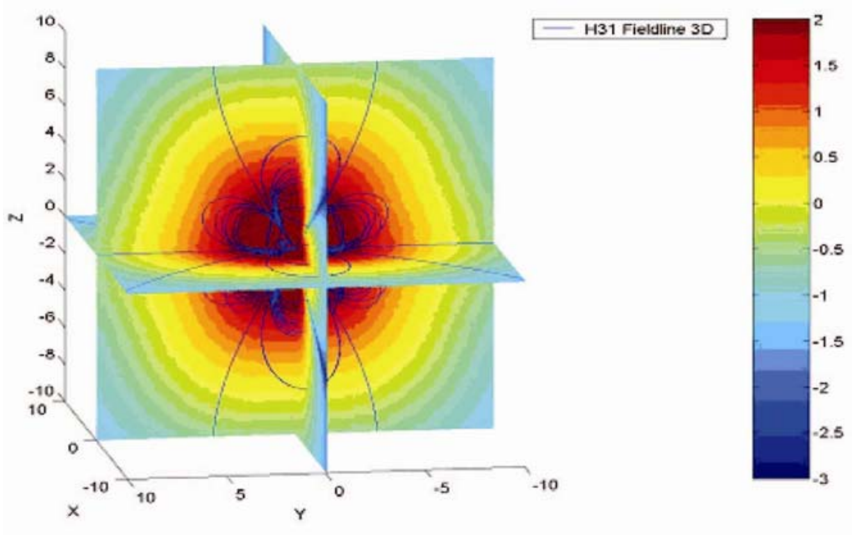


Octupole Moment 1

- While it is possible to decompose octupole moments into separated quadrupoles and separated dipoles, there is not a lot of physical insight gained in doing so.
- The g_{30} moment, as are all g_{n0} moments, is rotationally symmetric about the *z* axis.
- The g_{31} and h_{31} moments appear to be 90° rotations of each other. They are made of separated quadrupoles in the Y, X-Z planes.

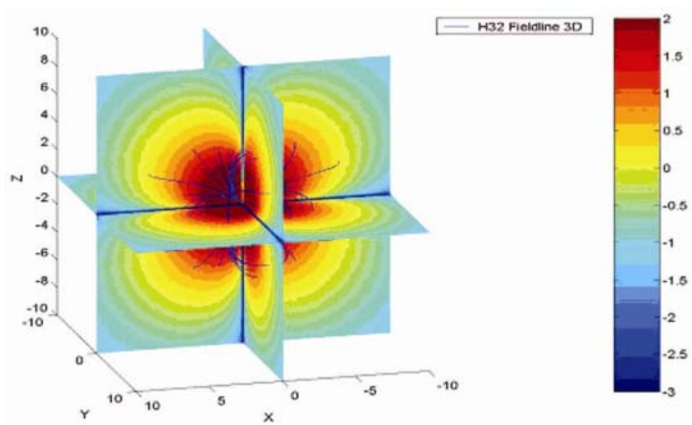


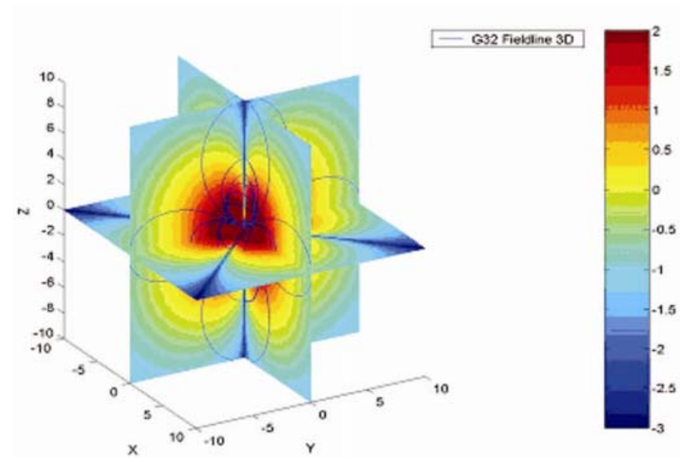


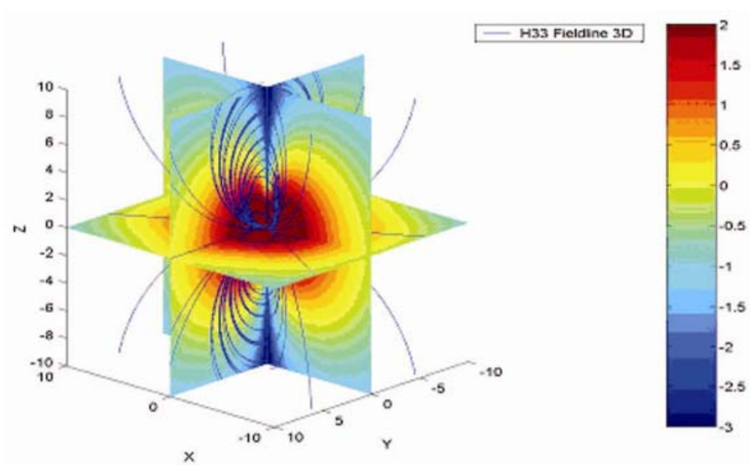


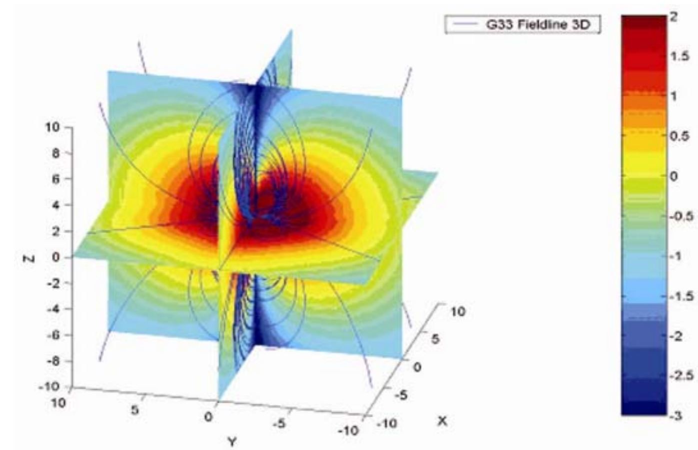
Octupole Moment 2

- The h_{32} and g_{32} moments have an interesting feature: three orthogonal directions of zero magnetic field strength. (Note contours are log scales.) Their quadrupole separations are all in a plane.
- The h_{33} and g_{33} are 'cubic' octupoles with their three quadrupole separations along each of x, y, and z.
- To learn more about magnetic moments, use the XSOLAR program









Inverting the Geomagnetic Field 1

- The equations on the right can be written in matrix notation as

  $y = Ax$

- The values of *y* are the observed magnetic field readings; the values of *x* are the amplitudes of all the multipole moments and the *matrix A* is constructed from the location of all the measurements.

- The singular value decomposition method (SVD) [Lanczos, 1971] is generally used to solve this problem. It rewrites the equation as

  $y = uSV^T x$

  where *U* is an *N x M* matrix; *S* is an *M x M* diagonal matrix of the singular values; V^T is a *M x M* matrix.

- The solution is

  $x = VS^{-1}U^T y$

- C. Lanczos, Differential operations, 564pp., D. Von Nostrand, Princeton, N.J. 1971.

$$
\begin{pmatrix} B_{r1} \\ B_{\theta 1} \\ B_{\phi 1} \\ \vdots \\ B_{ri} \\ B_{\theta i} \\ B_{\phi i} \\ \vdots \\ B_{rn} \\ B_{\theta n} \\ B_{\phi n} \end{pmatrix}
=
\begin{pmatrix}
X_{r11} & X_{r12} & \cdots & X_{r1m} \\
X_{\theta 11} & X_{\theta 12} & \cdots & X_{\theta 1m} \\
X_{\phi 11} & X_{\phi 12} & \cdots & X_{\phi 1m} \\
\vdots & \vdots & \ddots & \vdots \\
X_{ri1} & X_{ri2} & \cdots & X_{rim} \\
X_{\theta i1} & X_{\theta i2} & \cdots & X_{\theta im} \\
X_{\phi i1} & X_{\phi i2} & \cdots & X_{\phi im} \\
\vdots & \vdots & \ddots & \vdots \\
X_{rn1} & X_{rn2} & \cdots & X_{rnm} \\
X_{\theta n1} & X_{\theta n2} & \cdots & X_{\theta nm} \\
X_{\phi n1} & X_{\phi n2} & \cdots & X_{\phi nm}
\end{pmatrix}
\cdot
\begin{pmatrix} g_1 \\ g_2 \\ g_n \\ h_1 \\ \vdots \\ h_m \end{pmatrix}
$$

Inverting the Geomagnetic Field 2

- The *matrix* $A\ A^T$ is solved for its M largest eigen values and the associated eigen vectors.
- The *matrix* U is constructed with these eigen vectors as columns.
- The *matrix* S is a diagonal matrix consisting of the eigen values of A^TA and is assembled with the largest eigen value in the upper left. All elements are positive and decrease with size down and to the right.
- A measure of the stability of a linear system is the condition number, *CN*, defined as the ratio of the largest to the smallest singular value. The singular values are the square roots of the eigen values.
- Errors in the m^{th} generalized parameter can be expected to be about *CN* times larger than the errors in the first generalized parameter.
- Remember that *CN* depends only on the trajectory of the spacecraft if you are using spacecraft observations.
- Another quality indicator is the parameter accuracy indicator.

Inverting the Geomagnetic Field 3

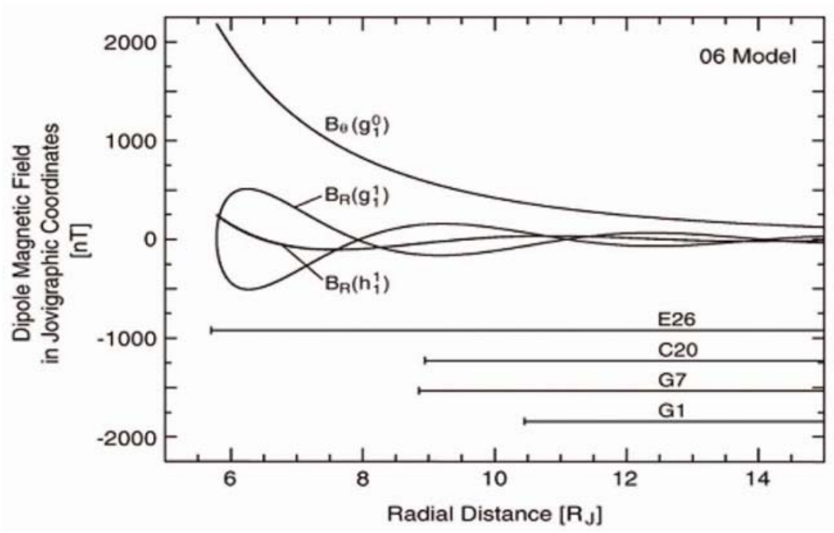
- Parameter accuracy for one parameter is defined as

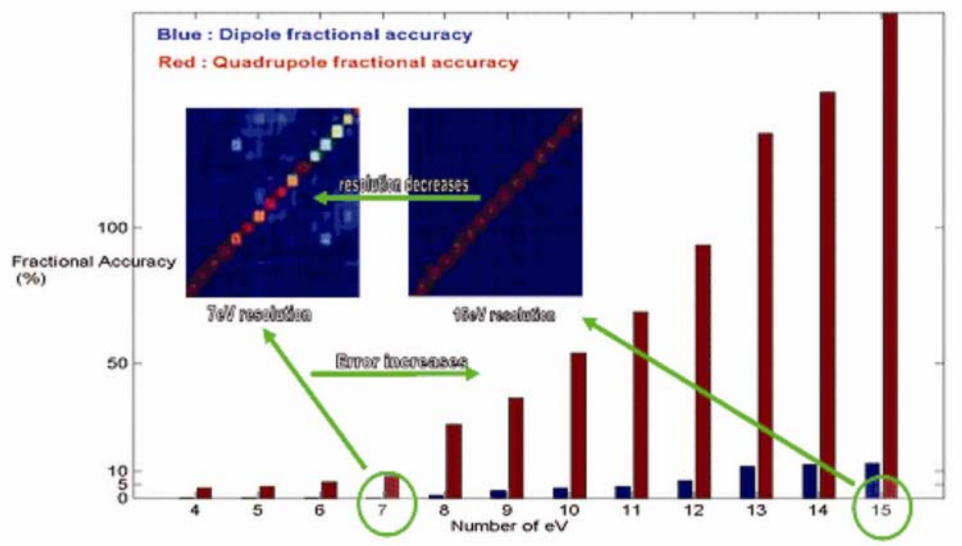
$$\delta(g,h) = \sum_{i=1}^{N} \frac{v_{j,i}^2}{sv_i^2}$$

- Fractional error for n^{th} multipole is based on knowledge of the observation noise level

$$\sigma_n = \frac{B_{noise}}{B_{nth}} \sqrt{\frac{\sum_m [(\delta g_n^m)^2 + (\delta h_n^m)^2}{2n+1}}$$

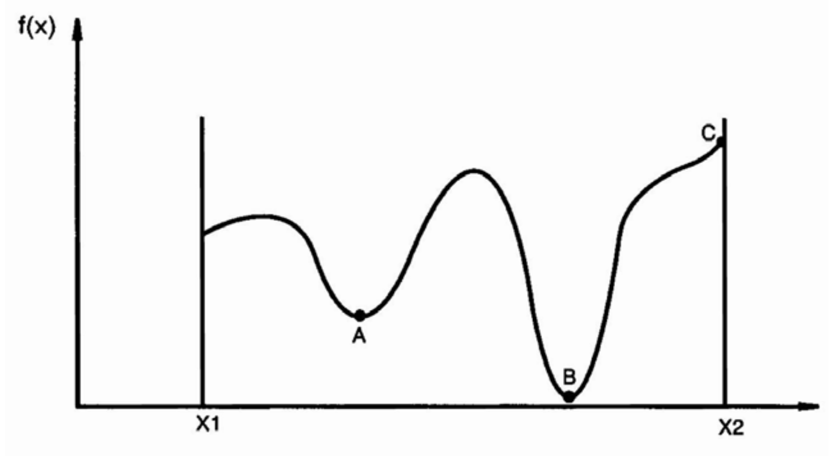
- There is a trade-off between the number of coefficients you can calculate and the resolution you can get

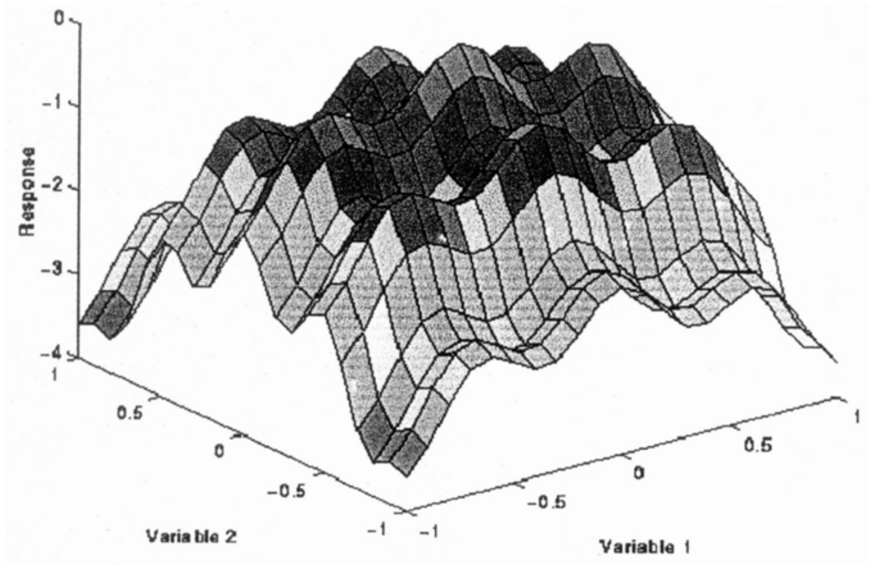




Downhill Simplex

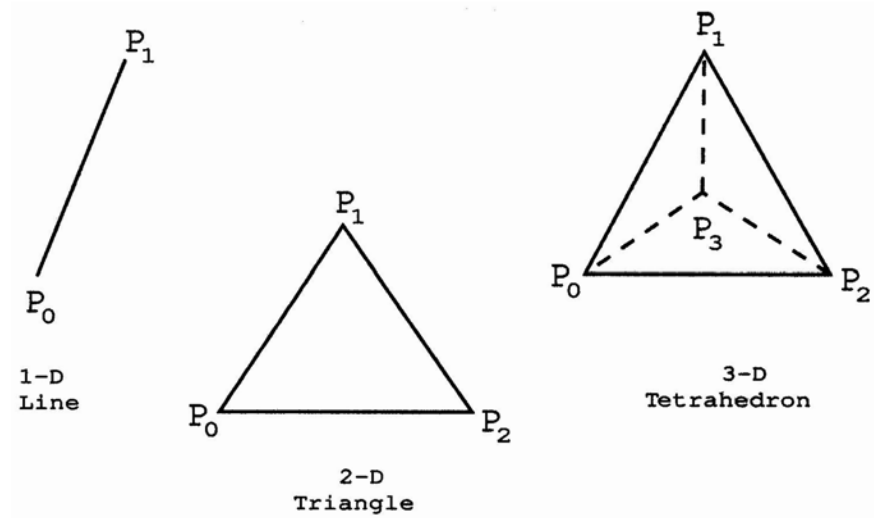
- The problem to be solved is to find a function f(x) that best matches a set of observations.
- As parameters are varied, one may read a local minimum. One is seeking the global minimum, so a zero-derivative is not necessarily the answer.
- Downhill simplex is a derivative-free method
  - Simple
  - Consistently goes downhill
  - Can oscillate near minimum
  - Basic downhill simplex has fixed steps
  - Modified downhill simplex has variable steps
  - Can be computationally expensive
- Multi-dimensional solutions can have many minima to track through

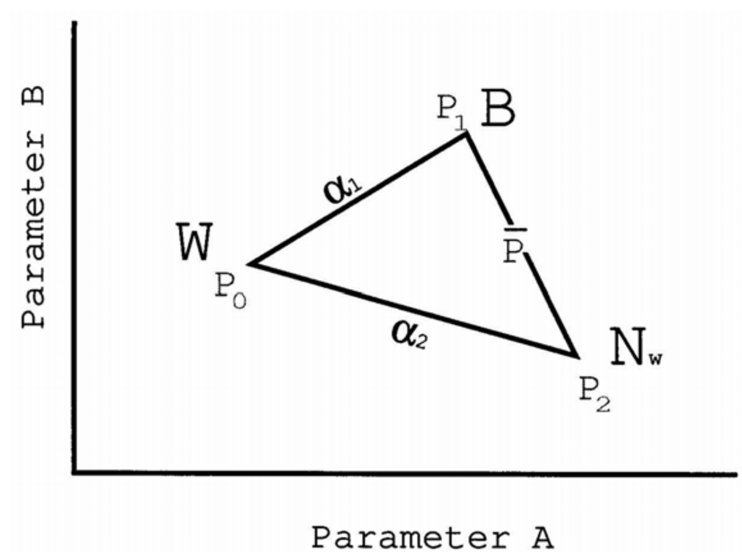




Definition of a Simplex

- A simplex is an object in *N* dimensions, consisting of the lines that join *N + 1* points
  - In one dimension a line
  - In two dimensions a triangle
  - In three dimensions a tetrahedron
- A degenerate simplex has colinear lines and infinite volume. A non-degenerate simplex encloses a finite volume.
- Initialization begins with *i + 1* trials, where *i* is the number of control parameters
  - Start with an initial guess, P_o, and step sizes in each dimension, e_i. The vertices of the simplex are at $P_i = P_o + e_i$
- Next, move the simplex according to 4 rules.





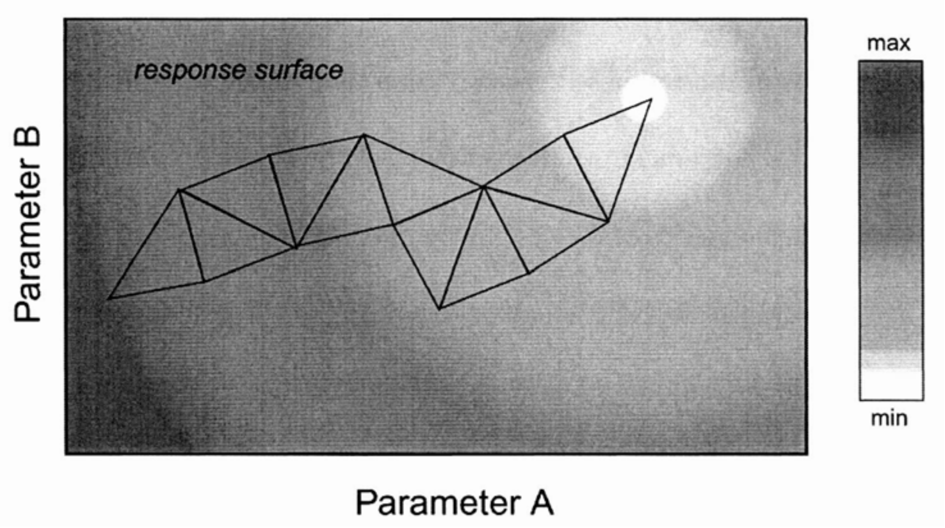
Rules of Movement

- Reject the trial with the least favorable response value in the current simplex and reflect the simplex away from that point.

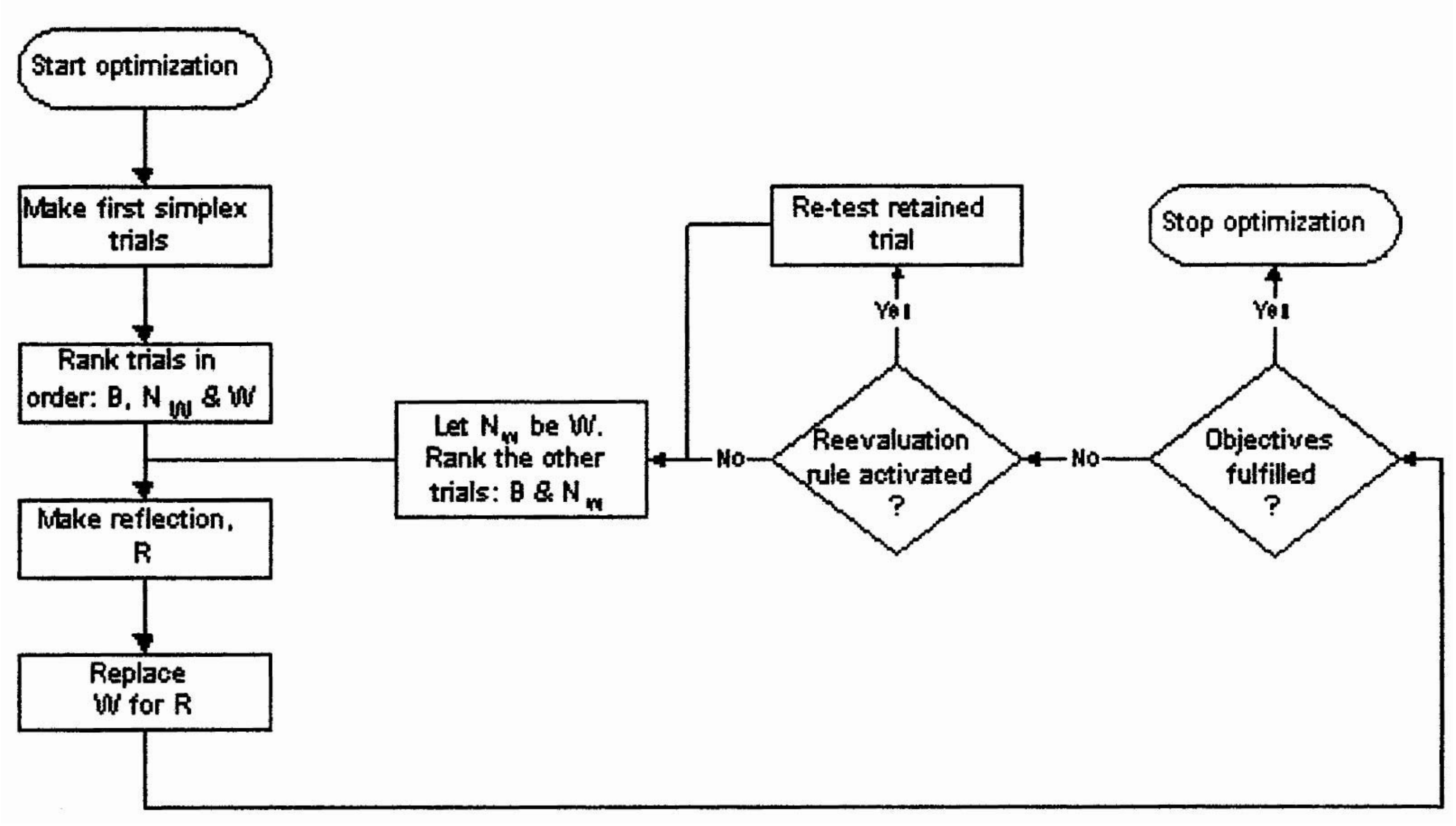
- Never return to control variable levels that have just been rejected so you do not oscillate in a local minimum.

- Reevaluate trials retained in the simplex for a specified number of iterations so you move away from poor local extrema.

- Do not accept trials calculated outside of established boundaries for the response function.

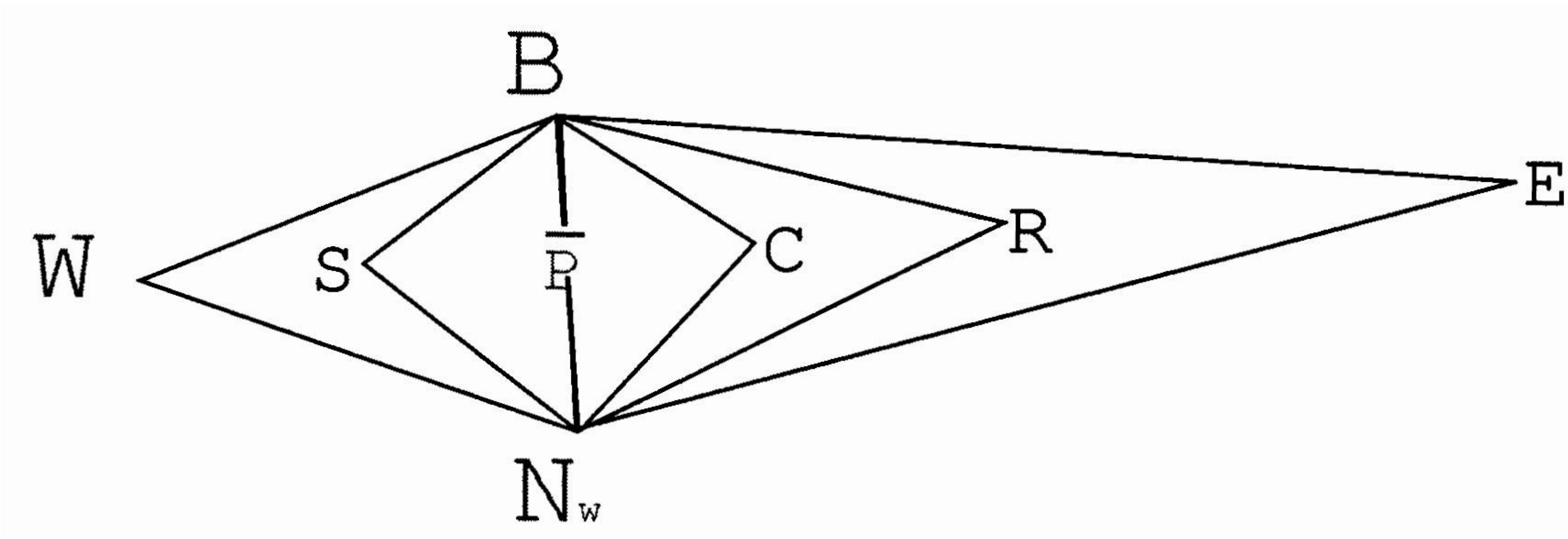


Basic Simplex Flowchart



- *B* is most favorable trial; *W* is worst and *NW* is next to worst.
- Move away from worst solution and re-rank trials.
- Repeat until satisfactory solution found.

Modified Simplex



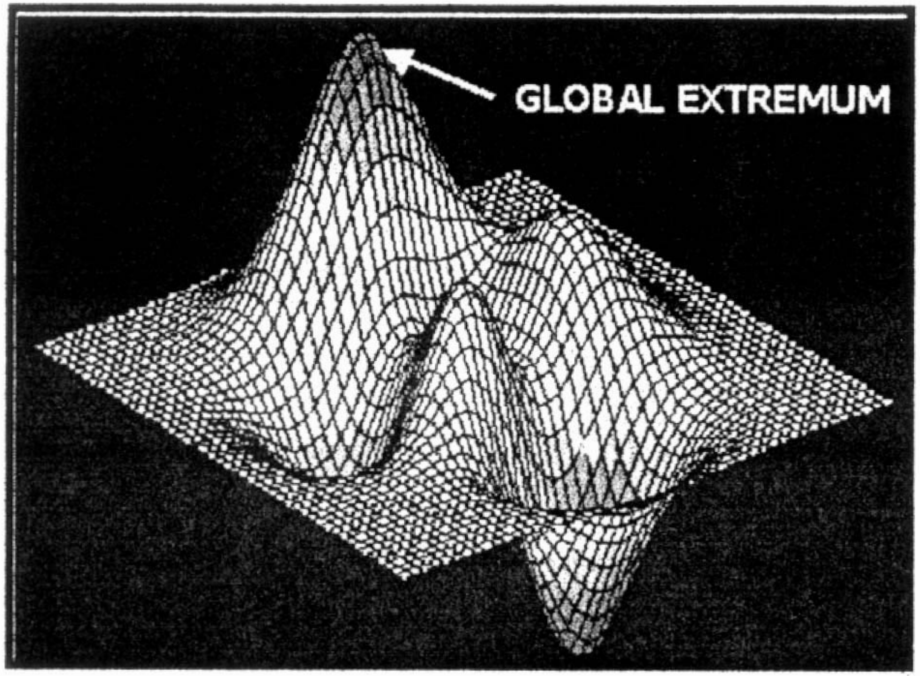
- Expand in a direction of more favorable conditions.

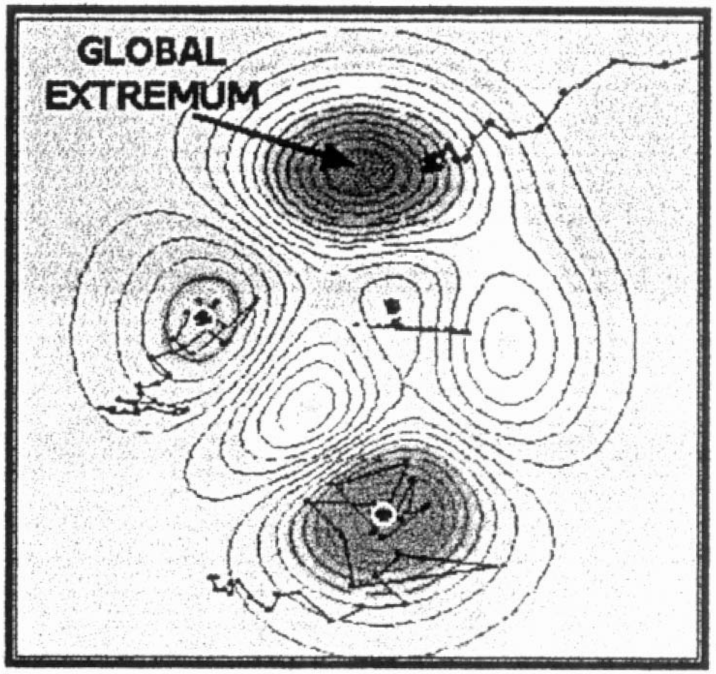
- Contract if a move was taken in a direction of less favorable conditions.

$R = \bar{P} + \alpha(\bar{P} - W)$ where $\bar{P}$ is average of remaining trials and α is reflection coefficient. E is expansion with larger coefficient.

- C is positive contraction; S is negative contraction.

Limitations of Simplex Technique

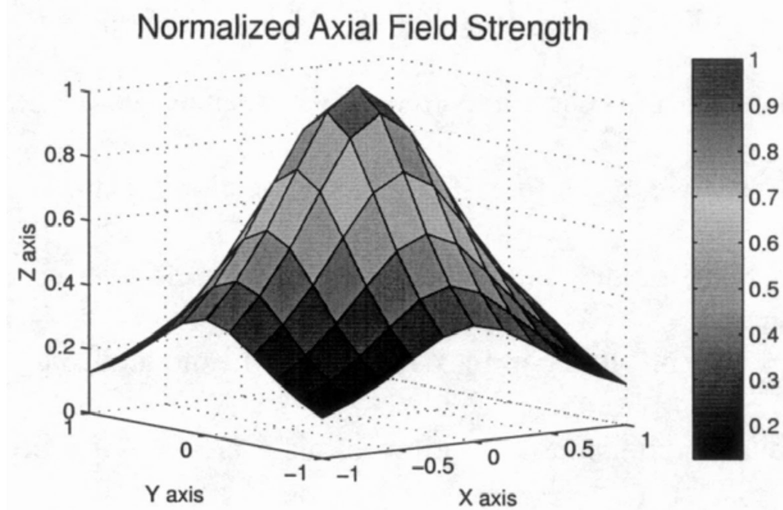


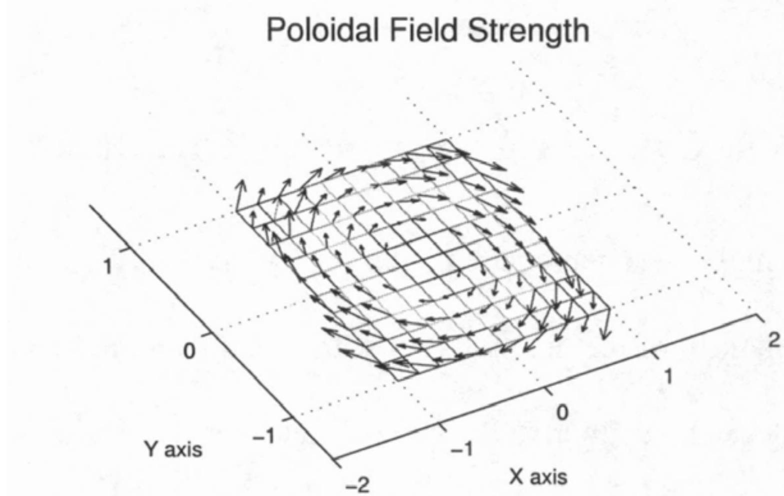


- Can find optimal variable settings for smooth response surfaces. Does not feature ability to move to the global optimum.

Application: Modeling of Magnetic Cloud Flux Ropes

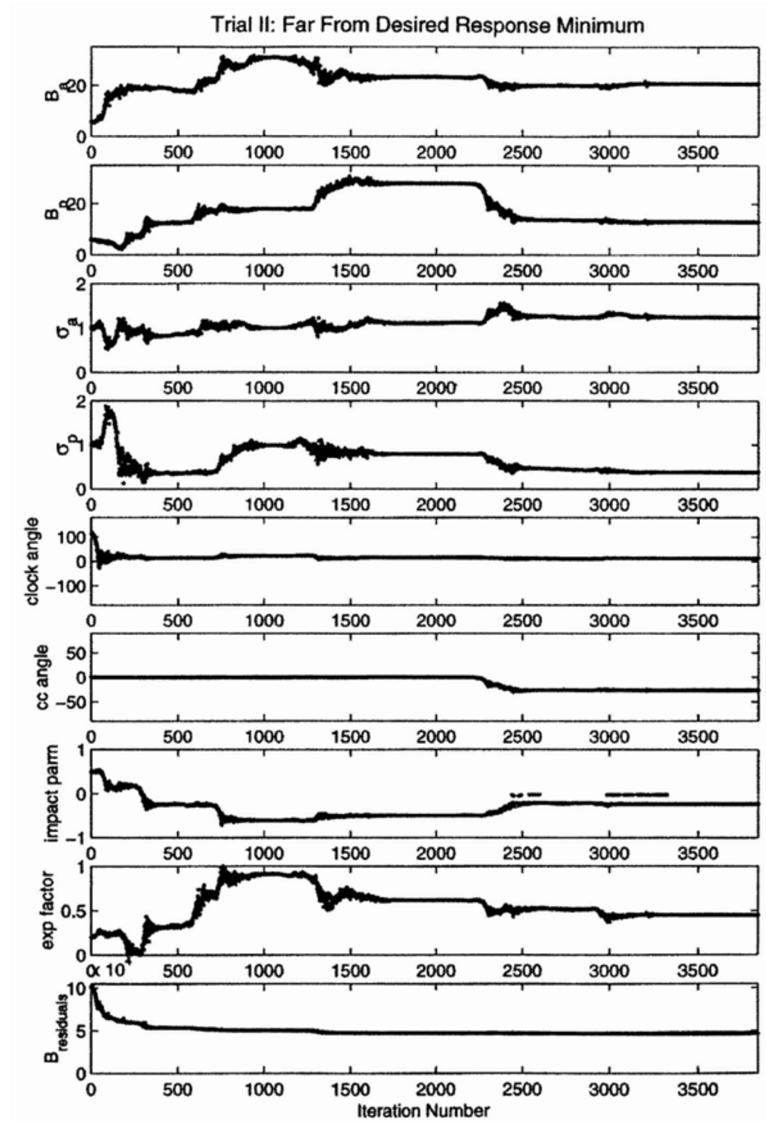
- Model a flux rope with gaussian distribution for magnetic field strengths. Rope can be right-handed or left-handed helix.
- Two field strengths: axial and poloidal.
- Two gaussian scale sizes.
- Two angles of axial orientation
- One impact parameter
- One expansion factor



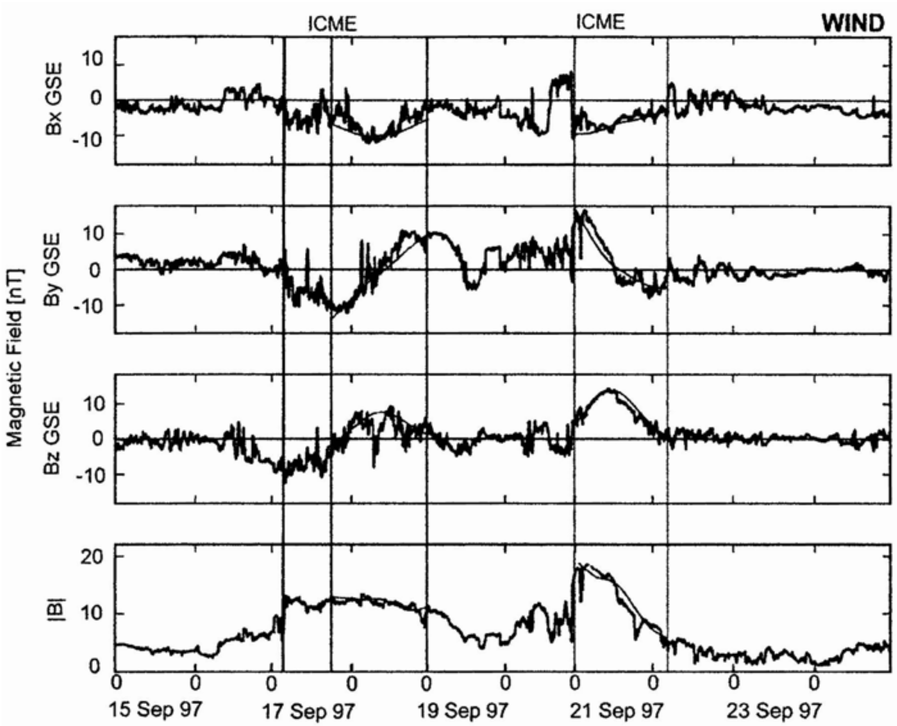


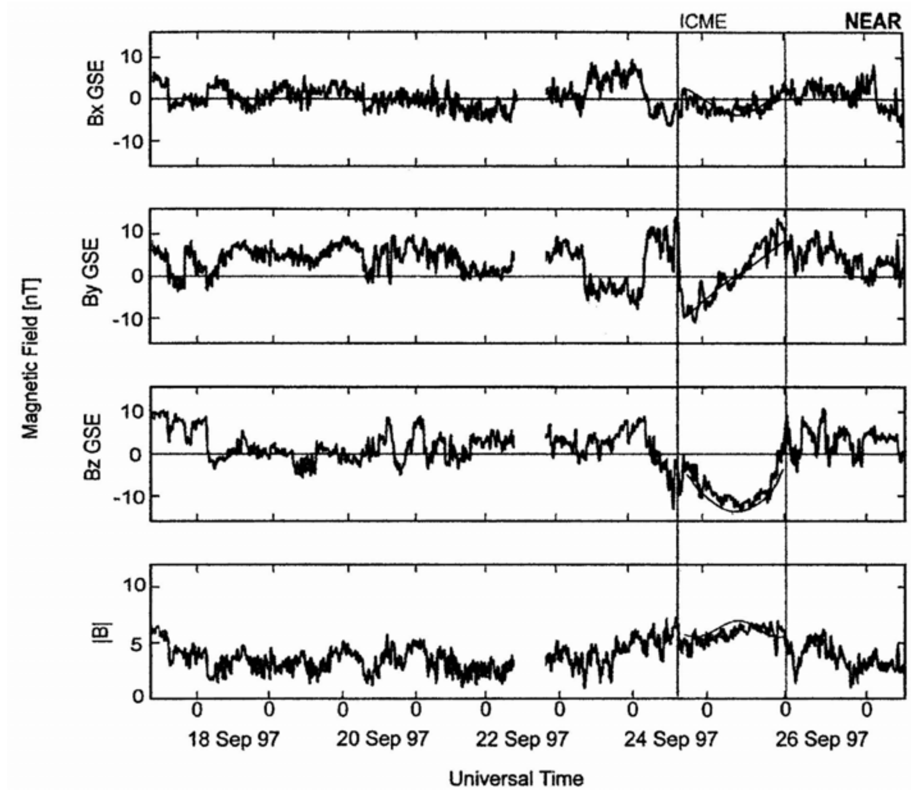
Evolution of Simple Inversion

- Initial trial is far from global minimum.
- Bottom trace is residual of fit.
- In course of nearly 4000 interations, solution changed greatly, often staying near one solution for a substantial period.



Sample Results





- Here are ICMEs found by Wind and NEAR and fit by this technique (Mulligan et al, 1999)

Summary

- There are a variety of ways to obtain the optimum mathematical fit to a set of data.
- The Singular Value Decomposition method is a common technique for planetary magnetic fields, and has been successfully used for Mercury, Earth, Jupiter and Saturn, but there are other techniques available.
- The Downhill Simplex technique avoids differentiating and uses only forward calculations.
- It has been successfully used to model the Io neutral torus and to model magnetic flux ropes. There are other techniques to model ropes including the Grad-Shafranov method.
- Do not be afraid to use these techniques.